

PROJECTIONS OF ZERO-SETS (AND THE FINE UNIFORMITY ON A PRODUCT)

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1. Introduction. We present here some results asserting that, under certain conditions on the pair of topological spaces (X, Y) , the projection π_X of $X \times Y$ onto X is “z-closed”, i.e., carries zero-sets onto closed sets. These results are intended to contribute to the description of the fine uniformity on a product space, *via* the following. (Proof in §6, see also [N, 1.6].)

1.1. *The semi-uniform product $X * Y$ of fine uniform spaces X and Y is fine iff π_X is z-closed.*

(The terminology on uniform spaces follows [I₁]. We consider only completely regular Hausdorff spaces. A zero-set is the set of zeros of a real-valued continuous function.)

For comparison with our results, we state the following theorem, due to Isbell, using results and methods of Glicksberg, Frolík, and Onuchic.

1.2 [I₁, Chapter VII]. *The uniform product of two fine uniform spaces is fine iff either (a) for some cardinal n , one factor is discrete of power $\leq n$, and the other is n -discrete, or, (b) for some cardinal n , the product is pseudo- n -compact and m -discrete for all $m < n$.*

(Some of these terms are defined below.)

Since the uniformity of the semi-uniform product is finer (larger) than that of the uniform product, each set of conditions in 1.2 is sufficient that π_X be z-closed. (In fact, see [N, 1.7].)

We point out explicitly that we have not obtained a complete classification of circumstances under which π_X is z-closed. See §5 for a discussion of this problem, and for remarks concerning the presumably simpler question of when π_X is closed. The latter, too, has not been completely answered although many results have been obtained (e.g., [HM], more extensively [N], [FF] and the references given there; here, 3.4 and §5.)

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2. Preliminaries. We first define some of the terms used in 1.2.

Let n be a cardinal number. A space is n -discrete [I₁, p. 135] if each intersection of n or fewer open sets is open. (It seems that these spaces were studied first by Sikorski [S], whose terminology differs.) Thus, each space is n -discrete for finite n , and an \aleph_0 -discrete space is a P -space [GJ, 4J].

Let n be an infinite cardinal. If in a space each locally finite family of open sets has power $< n$, the space is called pseudo- n -compact. (This is equivalent to the definition in [I₁, p. 135].) Thus, pseudo- \aleph_0 -compact = pseudocompact [G].

The following result of Noble's tells us, roughly, where to look for z -closed projections.

2.1 [N]. *If π_X is z -closed, then for each infinite cardinal n , either Y is pseudo- n -compact or X is n -discrete.*

(2.1 for $n = \aleph_0$ was observed earlier in [CN, 2.1] and [HM (b)].) The converse of 2.1 is (very) false. There is pseudo-compact Y and \aleph_0 -discrete X with π_X not z -closed ([CN, 4.6] and [HM (b)]).

So the sequel can be described briefly: we obtain some sets of conditions sufficient that π_X be z -closed, which sets are minimal, in some sense; the conditions will be stronger than those in 2.1, and weaker than those in 1.2.

3. From the example following 2.1, it follows that pseudocompactness of Y does not imply that π_X is z -closed. But compactness of Y does, of course. The following definition is involved in a generalization of this fact.

For n an infinite cardinal, a space is *weakly- n -compact* if each open cover has a subfamily of power $< n$ with dense union. This notion was introduced by Frolík [F₁] under a different name.

(For example, weakly- \aleph_0 -compact = compact; separable, and Lindelöf, spaces are weakly- \aleph_1 -compact. In general, a weakly- n -compact space is pseudo- n -compact; and for paracompact spaces, both these notions coincide with: each open cover has a subcover of power $< n$.)

3.1 THEOREM [HM (b)]. *If Y is weakly- n -compact, then whenever X is m -discrete for each $m < n$, π_X is z -closed.*

Proof. If $n = \aleph_0$, Y is compact and the conclusion follows. So we suppose $n > \aleph_0$, and that the hypotheses above are satisfied.

Let Z be the zero-set of f , and let $x_0 \notin \pi_X[Z]$. Let k be a positive integer. For each $y \in Y$, choose a neighborhood $U_y^k \times V_y^k$ of (x_0, y) on which f varies less than $1/k$. $\{V_y^k : y \in Y\}$ has a subfamily, $\{V_{y_\alpha}^k\}$, of power $< n$, with union dense in Y . Then $U_k = \bigcap_\alpha U_{y_\alpha}^k$ is an open set in X .

Let $U = \bigcap_{k=1}^\infty U_k$. Because $n > \aleph_0$, U is an open set containing x_0 ; and $U \cap \pi_X[Z] = \emptyset$, as we now verify.

Actually, for each $y \in Y$, the function of x , $f(\cdot, y)$, is constant on U . For, let $y \in Y$ and $x \in U$. Given $\varepsilon > 0$, choose neighborhoods $U_1 \times V$ and $U_2 \times V$ of (x, y) and (x_0, y) , respectively, with f varying less than $\varepsilon/3$ on each. Choose k with $1/k < \varepsilon/3$, and then choose $y' \in V \cap \bigcup_{\alpha} V_{y_{\alpha}}^k$. Now,

$$|f(x, y) - f(x_0, y)| \leq |f(x, y) - f(x, y')| + |f(x, y') - f(x_0, y')| + |f(x_0, y') - f(x_0, y)|.$$

The first and third terms are each $< \varepsilon/3$ by the definition of $U_1 \times V$ and $U_2 \times V$, respectively. The second term is $< \varepsilon/3$ because (x, y') , (x_0, y') both lie in the same $U_{y_{\alpha}}^k \times V_{y_{\alpha}}^k$.

The next result establishes that weak- n -compactness in 3.1 is the optimal condition. (See the remark following 3.2.)

3.2 THEOREM [HM (b)]. *Suppose that Y has the property that, for every space X which is m -discrete for all $m < n$, π_X is z -closed.*

(a) *If n is regular (i.e., not the sum of fewer than n smaller cardinals), then Y is weakly- n -compact.*

(b) *If n is not regular, then Y is weakly- n^* -compact, where n^* is the least cardinal larger than n .*

REMARK. The converse of 3.2 (b) is included in 3.1. For, if n is not regular, and X is m -discrete for all $m < n$, then X is n -discrete [I₁, p. 134]. Thus, for such X , if Y is weakly- n^* -compact, π_X is z -closed by 3.1.

Proof of 3.2. We first note that (a) implies (b). If the hypotheses in (b) are satisfied, then, from the Remark above, π_X is z -closed for every X which is m -discrete for all $m < n^*$. But n^* is regular, so (a) applies. So we prove (a).

Suppose n is regular, and Y is not weakly- n -compact. Choose an open cover of Y with the property that no subfamily with power $< n$ has dense union, and let \mathcal{R} be the set whose elements are these subfamilies of power $< n$. Let $X = \mathcal{R} \cup \{p\}$, topologized as follows. Members of \mathcal{R} are isolated; given $A \in \mathcal{R}$, the set $\{p\} \cup \{B : B \supset A\}$ is a neighborhood of p . It is easily shown that X is Hausdorff. Moreover, X is normal, because given two disjoint closed sets, only one can contain p so the other must be open. Because n is regular, the intersection of $< n$ neighborhoods of p is again a neighborhood, and X is m -discrete for $m < n$. Now, a continuous function f from $X \times Y$ to the reals is defined as follows. Given $A \in \mathcal{R}$, choose $y_A \notin \text{cl} \bigcup \{U : U \in A\}$. Let f_A be a continuous function on Y with $f_A(y_A) = 0$, and $f_A(y) = 1$ if $y \in \text{cl} \bigcup \{U : U \in A\}$. Let $f(p, y) = 1$ for all $y \in Y$, and $f(A, y) = f_A(p)$. With Z the set of zeros of f , it is clear that $\pi_X[Z] = X - \{p\}$, and that this set is not closed. Finally, f is continuous. This is automatic at each (A, y) . Given (p, y) , pick $A \in \mathcal{R}$ with y lying in a member U of A . On $(\{p\} \cup \{B : B \supset A\}) \times U$, f has value 1, so that f is continuous at (p, y) . The proof is complete.

4. In this section we establish a result which stands in good analogy with the sufficiency in 1.2(b). The result generalizes the sufficiency part of the following theorem of Tamano.

4.1 [T₁]. Let X and Y be pseudocompact spaces. π_X is z -closed iff $X \times Y$ is pseudocompact.

Tamano's proof of the sufficiency uses the Glicksberg theorem, that $\beta(X \times Y) = \beta X \times \beta Y$ if $X \times Y$ is pseudocompact [G]. A direct proof can be fashioned from an argument of Frolík [F₂, proof of 1.3]. The proof below generalizes this method.

4.2 THEOREM. If $X \times Y$ is pseudo- n -compact, and X is m -discrete for each $m < n$, then π_X is z -closed.

Proof. Let the hypotheses be satisfied. Let Z be the zero-set (in $X \times Y$) of f . Suppose $x \notin \pi_X[Z]$. For k a positive integer, let $Y_k = \text{cl} \{y : |f(x, y)| > 1/k\}$. Then, $Z_k = Z \cap (X \times Y_k)$ is a zero-set in $X \times Y_k$, $Z = \bigcup_{k=1}^{\infty} Z_k$, and $\pi_X[Z] = \bigcup_{k=1}^{\infty} \pi_X[Z_k]$. We shall show that, for each k , $x \notin \text{cl} \pi_X[Z_k]$. If $n > \aleph_0$, it follows that $\text{cl} \bigcup_{k=1}^{\infty} \pi_X[Z_k] = \bigcup_{k=1}^{\infty} \text{cl} \pi_X[Z_k]$, so that $x \notin \text{cl} \pi_X[Z]$. If $n = \aleph_0$, then the function on Y , $f(x, \cdot)$, is bounded away from 0 [GJ, 1G.2]; so for some k , $Y_k = Y$ and $x \notin \text{cl} \pi_X[Z]$ follows.

From the definition, it follows that an open subset of a pseudo- n -compact space has pseudo- n -compact closure. So each $X \times Y_k$ is pseudo- n -compact. It now suffices to prove

4.3. Let $X \times Y$ be pseudo- n -compact, and let X be m -discrete for all $m < n$. Let Z be the zero-set of f . If, on $\{x\} \times Y$, f is bounded away from 0, then $x \notin \text{cl} \pi_X[Z]$.

Suppose for simplicity that $f \geq 0$. (Otherwise consider $|f|$.) Let the hypotheses in 4.3 hold, but suppose $x \in \text{cl} \pi_X[Z]$. We generalize Frolík's inductive argument [F₂, 1.3], doing transfinite induction over the ordinals $< \omega_n$ = the least ordinal of power n .

Let a be a positive real number with $f(x, y) \geq a$ for all $y \in Y$. We shall define, for each $\alpha < \omega_n$, $(x_\alpha, y_\alpha) \in Z$, and open neighborhoods $W_\alpha = U_\alpha \times V_\alpha$, $W'_\alpha = U'_\alpha \times V_\alpha$ of (x_α, y_α) , (x, y_α) , respectively, such that $U_\alpha \subset \bigcap \{U'_\beta : \beta < \alpha\}$, and $f|W_\alpha \leq a/3$, $f|W'_\alpha \geq 2a/3$.

Choose $x_0 \in \pi_X[Z]$ and y_0 with $(x_0, y_0) \in Z$. Choose neighborhoods $W_0 = U_0 \times V_0$, $W'_0 = U'_0 \times V_0$, of (x_0, y_0) and (x, y_0) respectively, with $f|W_0 \leq a/3$ and $f|W'_0 \geq 2a/3$.

Let $\alpha < \omega_n$, and suppose that for each $\beta < \alpha$, (x_β, y_β) , $W_\beta = U_\beta \times V_\beta$ and $W'_\beta = U'_\beta \times V'_\beta$ have been defined which have the properties mentioned. There are fewer than n of the open sets U'_β , so $\bigcap \{U'_\beta : \beta < \alpha\}$ is open. Choose $x_\alpha \in \bigcap \{U'_\beta : \beta < \alpha\} \cap \pi_X[Z]$, and then y_α with $(x_\alpha, y_\alpha) \in Z$. Now choose $W_\alpha = U_\alpha \times V_\alpha$, $W'_\alpha = U'_\alpha \times V_\alpha$, neighborhoods of (x_α, y_α) , (x, y_α) , respectively, with $U_\alpha \subset \bigcap \{U'_\beta : \beta < \alpha\}$, and $f|W_\alpha \leq a/3$, $f|W'_\alpha \geq 2a/3$. This completes the induction step.

By pseudo- n -compactness, the family $\{W_\alpha : \alpha < \omega_n\}$ cannot be locally finite, so there is a point (\bar{x}, \bar{y}) with each neighborhood meeting infinitely many W_α . Evidently, this implies $f(\bar{x}, \bar{y}) \leq a/3$. But also, $f(x, y) \geq 2a/3$, because each neighborhood of (x, y) meets infinitely many W'_α as well. For, if $U \times V$ is a neighborhood of (\bar{x}, \bar{y}) , choose a countable infinity of ordinals $\alpha_1 < \alpha_2 < \dots (< \omega_n)$ with $W_{\alpha_i} \cap (U \times V) \neq \emptyset$ for each i . Since $U \cap U_{\alpha_{i+1}} \neq \emptyset$, it follows that $U \cap U'_{\alpha_i} \neq \emptyset$. Thus, $W'_{\alpha_i} \cap (U \times V) \neq \emptyset$, for each i .

We have a contradiction, and the proof is complete.

We will discuss extensions of 4.2 shortly. First we mention a “converse” of 4.2, due to Noble, which generalizes the necessity in 4.1.

4.4 THEOREM. *Suppose X is m -discrete for each $m < n$, but not discrete, and suppose X and Y are pseudo- n -compact. If π_X is z -closed then $X \times Y$ is pseudo- n -compact.*

(Actually, Noble does not quite state 4.4, but he proves it [N, 3.4].)

The question arises of what conditions on the factors make a product pseudo- n -compact. For the case $n = \aleph_0$, much is known. See, for example [F₂, §3], [G], [SS], and some of the references in the latter two papers. Undoubtedly, many of the results for $n = \aleph_0$ can be generalized. We confine the present discussion to two simple remarks, the first of which disposes of the case omitted in 4.4, of discrete X .

4.5. Suppose X is discrete. When is $X \times Y$ pseudo- n -compact? Let $|X|$ denote the power of X , and p the least cardinal such that Y is pseudo- p -compact. Then: $X \times Y$ is pseudo- n -compact iff $|X| < n$ and either (a) $p < n$, or (b) $p = n$ and n is not the sum of fewer than $|X|$ smaller cardinals. The proof is obtained by tracing locally finite families in $X \times Y$ on the subsets $\{x\} \times Y$.

The following is obtained from 4.4 and 3.1.

4.6 COROLLARY. *Suppose X is m -discrete for each $m < n$, and pseudo- n -compact. If Y is weakly- n -compact, then $X \times Y$ is pseudo- n -compact.*

This generalizes the well-known fact that $X \times Y$ is pseudocompact if X is pseudocompact and Y is compact [GJ, 9.14]. (Actually, in 4.6 the discreteness hypothesis on X can be omitted.)

5. Some remarks. The discussion focuses around extensions and modifications of 4.2. It is rather clear that the condition that π_X be z -closed should have little to do with global properties of X . For example, with very minor alterations the Proof of 4.2 works if each point of X has a neighborhood G such that $G \times Y$ is pseudo- n -compact.

Another approach derives from a desire to assume only conditions on X and Y , and not *a priori* on $X \times Y$. This leads to the question: what property of X is necessary and sufficient that $X \times Y$ be pseudo- n -compact for each pseudo- n -compact Y ? For $n = \aleph_0$, this problem has been solved by Frolík [F₂, 3.6], and the generalization shouldn't be too difficult. But again, for just the conclusion that π_X be z -closed, local properties of X ought to suffice. We are led to the question

5.1. *What property of X is necessary and sufficient that for each pseudo- n -compact Y , π_X is z -closed?*

(An analogous question is answered by the results in §3.)

For $n = \aleph_0$, it might be possible to solve this problem by “localizing” the condition [F₂, 3.6.1]. But the following question, which should be essentially simpler, has not been answered completely: what property of X is necessary and

sufficient that for each countably compact Y , π_X is closed? This question was raised by Isiwata [I₂]. The best partial answer is due to Isiwata, and Franklin and Fleischer [FF]: it is sufficient that X be a subspace of some sequential space; and the converse is not known.

Returning to 5.1 for $n = \aleph_0$, it is sufficient that X be sequential. ([HM (b)]; recall 4.1 and compare [I₂, p. 142, 5 (c)].) I doubt that it suffices that X be a subspace of a sequential space. But each sequential space is a k -space, and that X be a k -space is sufficient [T₁, p. 229].

Next, we indicate a procedure whereby new results can be derived from known ones of a certain type. One starts, for example, with the theorem: π_X is z -closed if Y is pseudocompact and X is first-countable (from 4.1 and [I, p. 142, 5 (c)]), then, by simultaneously strengthening the hypothesis on Y and weakening that on X , one preserves the conclusion that π_X be z -closed. This can be done in various ways; we mention two examples (without proof).

5.2 PROPOSITION. π_X is z -closed if X has property $*$ (x is in the closure of A iff x is in the closure of a countable subset of A) and Y has the property: given a sequence $\{U_n\}$ of open sets, there is a compact set K such that for each n , $K \cap U_n \neq \emptyset$.

5.3 PROPOSITION [HM (b)]. π_X is z -closed if in X Tukey's n -phalanxes [T₂] determine the topology, and Y is n -pseudocompact in the sense of Kennison [K].

Finally, we make some remarks on the relation between the two questions: when is π_X z -closed? and when is it closed?

It seems to be the case that theorems concerning the condition that π_X be z -closed which involve hypotheses "only on the factors X and Y " (e.g., hypotheses such as 1.2 (a) as opposed to 1.2 (b)) have exact analogues for the condition that π_X be closed.

In particular, call a space n -compact if each open cover has a subcover of power $< n$. If in 3.1 and 3.2, " z -closed" is replaced by "closed", " $weakly$ - n -compact" by " n -compact" and " $weakly$ - n^* -compact" by " n^* -compact", then the resulting statements are true. The proofs are simplified versions of those above ([HM], see also [N, §2].)

Continuing in this vein, 5.3 remains true if " z -closed" is replaced by "closed" and " n -pseudocompact" is replaced by the condition that each open cover of power $\leq n$ has a finite subcover.

There also is an analogue of 5.2, which we amplify a bit.

5.4 PROPOSITION. *The following conditions on a space Y are equivalent.*

(a) *For any space X : if F is a closed subset of $X \times Y$ then $\pi_X[F]$ contains the closure of each countable subset.*

(b) *For every X satisfying $*$ of 5.2, π_X is closed.*

(c) *Y has the property: each countable subset has compact closure.*

(The proof of 5.4 is not difficult, and we omit it.)

In footnote 8 of [FF] Franklin and Fleischer raise a question which I interpret to be: what spaces Y have property 5.4 (a)? They note that compact spaces do, spaces which do are countably compact, and a space which does need not be sequentially compact. They ask for an example of a sequentially compact space without property 5.4 (a). Because of 5.4 (c), a separable sequentially compact space which is not compact would be an example. M. E. Rudin has constructed such a space, in [R], using the continuum hypothesis. (This was pointed out to me by A. H. Stone.)

The following is easy to prove.

5.5 PROPOSITION. *If π_X is z -closed and $X \times Y$ is normal, then π_X is closed.*

I know of no theorems with conclusion “ π_X is closed” without hypotheses solely on the factors X and Y , except some which follow from 5.4 and a theorem (like 4.1) with conclusion “ π_X is z -closed”.

6. In [I₁, III. 39] Isbell notes that the semi-uniform product $X * Y$ of fine uniform spaces X and Y is fine iff $X \times Y$ is C^* -embedded in $X \times \beta Y$ (i.e., each bounded continuous real-valued function on $X \times Y$ has a continuous extension over $X \times \beta Y$). This, and the following, suffice to prove 1.1.

6.1 PROPOSITION [HM (b)]. *π_X is z -closed iff $X \times Y$ is C^* -embedded in $X \times \beta Y$.*

Proof. Let π_X be z -closed. By [D, X.5.3], it suffices to show that, given $f, y_0 \in \beta Y - Y$, $x_0 \in X$, and $\varepsilon > 0$, there is a neighborhood G of (x_0, y_0) in $X \times \beta Y$ such that on $(X \times Y) \cap G$, f varies by less than ε . Since the function on Y , $f(x_0, \cdot)$, extends to y_0 , there is a neighborhood V of y_0 such that on $Y \cap V$, $f(x_0, \cdot)$ varies by less than $\varepsilon/3$. The zero-set $Z = \{(x, y) : |f(x, y) - f(x_0, y)| \geq \varepsilon/3\}$ has closed projection on X , so there is a neighborhood U of x_0 missing $\pi_X[Z]$. $G = U \times V$ is the desired neighborhood of (x_0, y_0) .

For the converse, recall that [GJ, 6.4] $X \times Y$ is C^* -embedded in $X \times \beta Y$ iff disjoint zero-sets in $X \times Y$ have disjoint closures in $X \times \beta Y$. Suppose π_X is not z -closed, and let f be a function whose zero-set Z has $x_0 \in \text{cl } \pi_X[Z] - \pi_X[Z]$. $g(x, y) = |f(x, y) - f(x_0, y)|$ defines a continuous function; let Z_1 be its zero-set. Evidently, $Z \cap Z_1 = \emptyset$. Let a bar denote closure in $X \times \beta Y$. We have $(\{x_0\} \times \beta Y) \cap \bar{Z} \neq \emptyset$, and $\{x_0\} \times \beta Y \subset \bar{Z}_1$. So $\bar{Z} \cap \bar{Z}_1 \neq \emptyset$, as desired.

(Independently, Comfort and Negrepontis have shown that $X \times Y$ is C^* -embedded in $X \times \beta Y$ if π_X is closed [CN, 3.1]. Their proof is the same as that above.)

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